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The Opportunities for Mathematical Reasoning Skills That Teachers Provide for Their Students in The Learning Environment

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The acquisition of mathematical reasoning skills is directly related to the opportunities gained by the students in the learning environment. The complete lack or the rarity of such situations that support mathematical reasoning in the learning environment is an obstacle to conceptual learning. Therefore, it is important to investigate the extent to which different aspects of reasoning are encountered in teachers' presentations and to convey the reflections of those from the learning environment. The objective of this study is to examine the opportunities for mathematical reasoning skills that teachers give to their students. A holistic case study design was used in the study process as a qualitative research approach. The study was carried out with two teachers working in public schools. Unstructured observations and video recordings conducted by the researchers were used to collect data, and content and descriptive analysis analyse the obtained data. When the teachers were evaluated regarding the opportunities for mathematical reasoning they provided in their classes, it was concluded that although these constitute a variety of opportunities, they present limited opportunities that could fully support mathematical reasoning. It is thought that offering limited opportunities in the learning environment may negatively affect students' mathematical reasoning skills.

Introduction

Mathematical reasoning can be considered as the ability to draw conclusions about a particular subject, as a problem solving tool or as a general ability (Hjelte, Schindler & Nilsson, 2020). According to Altıparmak and Öziş (2005), proof and reasoning is an instinctive human ability, and the development of this ability depends on appropriate strategies. In fact, failure to determine these strategies in a purposeful manner blunts these innate abilities and leads to the upbringing of individuals who cannot follow the cause and effect relationship and prefer memorization. When students do not reason, they often fail to understand that different ideas in mathematics are interrelated and dangerously come to

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believe that mathematics is just a set of facts and methods to be remembered (Boaler, 2010). Mathematical knowledge refers to the numerical solution that can be related to a memorized set of mathematical facts, such as an addition table, while mathematical reasoning is an argument made by the solver to justify a mathematical process, procedure, or conjecture (Mevarech & Iddini, 2021). According to Işık (2007), mathematics should not be left in students' minds as a series of complex methods and formulas that are perceived as too difficult to remember and memorize; it should be emphasized that mathematics is based on thinking, reasoning, and intuition. The National Council of Teachers of Mathematics (NCTM) (2000) states that developing students' reasoning is based on certain assumptions and rules and that students should be encouraged to justify and make assumptions.

This research is concerned with the opportunities for mathematical reasoning skills offered to students in learning environments.

Theorical Framework

Mathematical reasoning refers to drawing conclusions about certain ideas based on facts attained through logical and critical thinking in solving mathematical problems (Rohana, 2015). Reasoning is a very important aspect of mathematical skills in both learning and teaching mathematics (Sukirwan, Darhim, & Herman, 2018). According to Altiparmak and Ozis (2005), reasoning is not only a mathematical but also a fundamental skill, and there is a close relationship between the development of reasoning skills and programs implemented in schools. Since the behaviours associated with mathematics are relevant at all levels and in all fields from primary and even preschool education programs to higher education programs (Baykul, 2014), it is necessary to start improving the basic skills of mathematical reasoning from early ages. Students who complete their primary education with appropriate mathematical reasoning-oriented mathematics training will be able to express their thoughts and will be eager to investigate the problems they face (Russell, 1999).

Five process standards are specified in the standards of the National Council of Teachers of Mathematics (2000). These are problem-solving, communication, association, reasoning with representation, and proof. Also, In Türkiye, the middle school mathematics curriculum (2013) includes also reasoning within the scope of mathematical process skills. In the Ministry of National Education (2013), reasoning is defined as “the process of obtaining new knowledge based on the available information, using the tools (symbols, definitions, correlations, etc.) and thinking techniques (induction, deduction, comparison, generalization, etc.) specific to mathematics”. In Ministry of National Education (2018), among the specific objectives of the mathematics curriculum, it is stated as "Students will be able to express their thoughts and reasoning easily in the problem-solving process and will be able to see the deficiencies or gaps in the mathematical reasoning of others". Therefore, the need for the preparation of environments intended to improve this skill in the mathematics teaching process is emphasized (Ministry of National Education, 2013). To allow students to express their opinions comfortably, a free environment must be provided where they can advocate their thoughts. Environments where thoughts are openly discussed without fear, different ideas are deemed important, and efforts are made to think together are of great importance for the emergence of different reasoning approaches (Umay, 2003).

The mathematical reasoning process requires a guide, such as a teacher (Brodie, 2010). Creating opportunities for students to be taught mathematical thinking also requires teachers to think mathematically (Stacey, 2006). Educators have a great responsibility in this regard.

Teachers trying to impose their own examples of reasoning on students, rather than acknowledging or understanding the various solution methods offered by students, miss out on rich opportunities that would enable their students to learn mathematics (House, 1999). In this sense, working with uniform reasoning in learning contexts may not lead to success for all types of reasoning (Lithner, 2005). The types of reasoning that students encounter in their learning environments will pave the way for them to adopt those types of reasoning more readily. It can be predicted that students will not seek alternative solutions or other ways of thinking if their teachers' methods of solving problems are presented to the students as the only acceptable, definitive approach. In such cases, students will tend to continue the process by taking their teachers as examples.

Reasoning needs conceptual support, and so mathematical activities such as implementing rote-learning procedures or offering memorized facts cannot be considered reasoning (Melhuish, Thanheiser, & Guyot, 2018). Therefore, efforts are required to improve mathematical reasoning skills. In this regard, teachers who aim to teach mathematics effectively (Tang & Ginsburg, 1999) have a great responsibility. Attitudes and behaviours such as teachers not showing tolerance to different questions from their students, providing inadequate answers to questions, expecting students to accept what was said without questioning, using limited assessment methods, or not including students in the learning environment have considerable effects on the development of reasoning (Kocagul Saglam & Unal Coban, 2018). Teachers should prepare environments that will allow for different ideas and support students from different cultural backgrounds and socioeconomic levels in being active in their classes because such environments are suitable for the development of mathematical reasoning (Umay, 2003). Therefore, effective mathematics education for students of various cultural backgrounds will reject the existence of a single culture or a uniform reasoning method in the learning process (Malloy, 1999). Since no person's characteristics and background are the same as those of another person, people's mental structures cannot be expected to be the same (Umay, 2003). When supportive environments are provided, all students can refute claims and engage in reasoning (Yackel & Hanna, 2003).

Widiartana (2018) stated that one of the causes of the poor quality of students' reasoning in mathematics is the fact that, in classroom learning, teachers are too oriented towards procedural aspects such as teacher-centered learning and mathematical concepts are often delivered to students as information without a deep understanding. Teachers should be aware of what their students can do to improve their mathematical reasoning skills and they can offer opportunities for the development of students' mathematical ideas by planning questions that reflect their reasoning and predictions (Olson, 2007). Teachers can develop students' logical thinking, judgment, and reasoning skills through simple questions and prompts, such as asking them why they did something in class, how they reached a decision, what conclusions can be drawn, what kinds of strategies were used to solve a problem, or what it would be like if a different approach were used, among other questions (Bahtiyari, 2010). Therefore, the complete lack or the rarity of situations that support mathematical reasoning in the learning environment is an obstacle to conceptual learning.

Importance of research and research problems

Mathematical reasoning is a crucial skill for achieving mathematics learning goals (Putra, Fauzi & Landong, 2020). The emphasis on mathematical reasoning in curriculum standards is part of an international trend, but identifying and understanding reasoning continues to challenge teachers (Davidson, Herbert & Bragg, 2019). In order to recognize, teach and assess reasoning, a more detailed knowledge of reasoning is required (Herbert, Vale, White & Bragg, 2022). One of the common reasons for students' inadequate mathematical reasoning is related to the teaching and assessment approaches adopted by teachers in their lessons (Mukuka, Mutarutinya & Balimuttajjo, 2021).

When the studies on reasoning skills are examined, it is seen that there are various studies on the development of mathematical reasoning skills and teachers' interest in directing their attention to the development of mathematical reasoning (Arnesen & Rø, 2022; Bergqvist & Lithner, 2012; Davidson, Herbert, & Bragg, 2019; Herbert & Bragg, 2021; Herbert, Vale, White & Bragg, 2022; Jäder, Sidenvall & Sumpter, 2016; Jeannotte & Kieran, 2017; Mata-Pereira & da Ponte, 2017; Mueller, Yankelewitz & Maher, 2014; Olsson & Granberg, 2022; Saleh, Prahmana, Muhammad & Murni, 2018; Xin, Chiu, Tzur, Ma, Park & Yang, 2020).

To plan a teaching process via which mathematical reasoning can be ensured, teachers must have knowledge about how to provide mathematical reasoning skills. Within this framework, the present study was undertaken to examine the opportunities for mathematical reasoning skills that teachers provide to students while considering the importance of identifying reflections from the teaching environment related to mathematical reasoning skills. In addition, it is important to investigate the extent to which different aspects of reasoning are encountered in teachers' presentations and to convey the reflections of those from the learning environment. In the present study, in order to investigate the opportunities for mathematical reasoning skills that teachers offer to their students:

- How are teachers' classroom teaching environments in terms of mathematical reasoning?
- What are the opportunities provided in terms of developing students' mathematical reasoning in the examples presented by teachers?

answers to the research questions were sought.

Method

Design of the Research

Within the scope of this study, a qualitative research approach was used with the intent to investigate the mathematical reasoning skill opportunities provided by primary mathematics teachers to their students. More specifically, a holistic case study was carried out. Case studies constitute a research method through which a current situation is described, or situational themes are created regarding a current phenomenon, real-life trends, or limited multiple situations (Creswell, 2013). Among the qualitative research approaches, the case study method was preferred for the present study because this research intended to reveal a current situation by examining the presentations of teachers in a teaching environment concerning mathematical reasoning skills, the research data were obtained through observations and video recordings, and humans were the focus of the study. Since the opportunities for mathematical reasoning skills offered to students by teachers in the field of algebra learning were dealt with holistically, the holistic multi-case design was selected for this study from among the possible case study designs. According to Yildirim and Simsek (2008), a study with a holistic multi-case design involves the holistic evaluation of each case within its own structure and the subsequent comparison of those cases with each other.

Participants

The research group comprised two teachers working in different schools. To select the teachers, teachers working in seven different schools were initially interviewed. Interviews were conducted with these teachers to form a volunteer-based working group and to see if the teachers were willing to further support the work. After the interviews, an appropriate sampling method was used to determine three teachers who were expected to contribute to the study and provide the opportunity to minimize the loss of time and labor. Of the two teachers that participated in the end, Eda had 12 years of teaching experience and Tulay had 7 years of teaching experience. Both teachers graduated from a primary school mathematics teacher education program. Classes with students who were considered average in terms of academic achievement, talkative and good communication skills were determined in line with the teachers' suggestions.

Pseudonyms were used instead of the real names of the participants in terms of the ethics of the research. The teachers' consent was obtained for the video recordings. All necessary permissions were obtained from the Provincial Directorate of National Education to carry out this study in the selected public schools.

Data Collection Tools and Implementation of the Study

Observations and video recordings were used as data collection tools in this study. The video recordings were made to avoid overlooking certain situations during the observations and to better describe the classroom environment. Observation is a method of collecting data used to observe the research subject, the events and processes occurring in the field, and the people involved in these events (Güler, Halıcıoğlu, & Tasğın, 2013). Observations and video recordings were chosen as data collection tools to make it possible to describe the interactions of students and teachers in the classroom and to allow opportunities to identify the mathematical reasoning skills that the teachers presented to the students. The researcher conducted these observations while the teachers delivered classroom units on equality and equations and on linear equations.

At the beginning of the research process, a pilot study was conducted with a teacher who was not included in the main research group in order to determine any problems that could occur in the process. During this pilot study, data were obtained by the researcher through observations, but during the analysis, it was found that some field notes were insufficient. For that reason, it was thought that there could have been parts that the researcher missed while conducting observations in the learning environment. In the main study, video recording was performed in addition to the observation of the classes to avoid this problem and facilitate better descriptions of the learning environment.

Data Analysis

The data obtained from observations during the research were subjected to content analysis and the data obtained from the video recordings were analyzed descriptively. Codes and categories were created by reviewing the observation data repeatedly. For the analysis of the video recordings, the data were first organized and separated into smaller units for analysis. Each presentation transcript was evaluated in terms of expressions used by the students, without considering the teachers' implicit intentions or purposes. The framework proposed by Bergqvist and Lithner (2012) was used to determine what had been observed during the teachers' presentations, in which they provided students the opportunity to apply different types of mathematical reasoning. Based on the draft produced in this process, the analysis of problems included in the teachers' presentations was performed by considering six perspectives, including the identification of the task type, recognizing a solution method, creative reflection, argumentation, the mathematical foundation, and alignment.

(1) Identification of the Task Type: This perspective was considered to determine whether or not the type or nature of the problem had been clearly defined by the teacher. A clear description would define the general characteristic features of the family of mathematical problems to which a particular problem belonged.
(2) Recognizing a Solution Method: This step has two sub-dimensions related to the selection and implementation of the strategy. It considers whether or not the teacher defines (a) the connection between the type of problem and the selection of strategy and (b) the principles related to the key components of the solution method. Defining the principles entails more than simply talking about the selection of a strategy (e.g., "this problem is solved by division") or explaining each step during the solution process. It does not necessarily involve any arguments; however, it requires defining not only a specific problem but also certain principles that apply to the problem type. Although this and the previous perspective stem from routine problem-solving properties, they also apply to types of creative reasoning that can be understood as differing from algorithm-based reasoning.
(3) Creative Reflection: An important distinction between creative reasoning and algorithm-based reasoning is that in creative reasoning, the selection and implementation of a particular strategy is not clear from the beginning, and metacognition may be necessary to avoid diversions and support the fluency and flexibility of the reasoning. The use of metacognition may involve questioning, analyzing, correcting inappropriate strategy selections or other errors, verifying information, or evaluating alternative solution strategies.
(4) Argumentation: The persuasiveness of choices and results can be considered in two ways, which include predictive argumentation and verificative argumentation. Predictive argumentation is clearly expressed prior to the conclusion, and if the reasoning only begins with the conclusion, predictive argumentation does not occur. Similarly, because the individual knows the result before reasoning, creative reasoning also does not occur. Verificative argumentation, on the other hand, arises in the form of explanations presented after the results are achieved, as may be seen during a teacher's presentation to confirm predictions. Efficient discussions along these lines can help students understand solutions achieved through creative reasoning, algorithm-based reasoning, and rote learning.
(5) Mathematical Foundation: Similarly to creative reasoning, mathematical reasoning depends on the real mathematical properties of the considered components. This perspective arises when results are based on the relevant properties. For example, "the result is correct because the components have the appropriate mathematical properties" reflects a mathematical foundation.
(6) Alignment: This perspective considers whether or not the solutions of the teachers are similar to those of the students. A lack of alignment may occur in this regard when teachers' reasoning processes are too challenging or inaccessible for students.

The Role of the Researcher

The video recordings and observations were made by the researcher to characterize the classroom environment. Before beginning this process of data collection, a few lessons were observed to ensure that the students and teachers acclimated to the researcher and behaved comfortably. The researcher sat at the back of the room and the video recordings were made with a camera that remained in a fixed position during the course to avoid affecting the learning environment or distracting the teacher and students. Following the completion of the data collection, the researcher edited the observation notes, documented the findings of the video recordings, prepared for the analysis, and conducted the analysis.

Plausibility, Transferability, Consistency, and Verifiability of the Study

To ensure the plausibility of the study, long-term interactions with the participants were ensured and diversification was sought with the use of multiple data collection tools. To ensure the transferability of the study, the stages of the research and the learning environment were characterized in detail. Consistency was achieved by using direct quotations and performing analyses to compare data with each other. Finally, verifiability was achieved by providing a detailed explanation of the analysis methods, an explanation of the researcher's role, and detailed descriptions of the participants and the acquisition and usage of all data including the data collection tools and video recordings.

Findings

In this section, the order of instruction of the subjects that were taught in the course of this research, the lesson hours allocated to each subject, and the content analysis of the observation results are first presented. Subsequently, dialogues and presentations from the classroom environment during the solving of the examples given by the teachers are analyzed in detail. The analyses of the teachers' presentations were performed by applying six perspectives, which included the identification of the task type, recognizing a solution method, creative reflection, argumentation, the mathematical foundation, and alignment.

Findings Regarding the Learning Environments in the Teachers' Classrooms

In the learning environments in the classrooms of Eda and Tulay, the subjects of equality and equations and linear equations were taught in the order given in Table 1.

Table 1. Order of instruction applied by Eda and Tulay for teaching the subjects of equality and equations and linear equations

Eda		Tulay	
Time	**Subjects of the lessons**	**Time**	**Subjects of the lessons**
2 lessons	Writing the algebraic expressions corresponding to verbal expressions	2 lessons	Algebraic expression, equation, unknown concepts
2 lessons	Equality and equations	2 lessons	Equation posing/solving problems
5 lessons	Equation solving	6 lessons	Equation solving
5 lessons	Equation posing problems	2 lessons	Equation posing problems
4 lesson	Coordinate planes	2 lessons	Linear equations
3 lesson	Linear relationships	8 lessons	Line graphs
4 lesson	Line graphs		

As shown in Table 1, the teaching approaches of these teachers differed from each other. They followed almost the same order of instruction in terms of teaching the subjects of equality and equations and linear equations as both teachers allocated 12 lessons for the teaching of equality and equations. However, Eda allocated 11 lessons for teaching the subject of linear equations, whereas Tulay allocated 8 lessons. Eda initially reminded students about how algebraic expressions correspond to "various verbal expressions." On the other hand, Tulay introduced the topic by telling students about the concepts of "unknowns, equations, and variables” within the scope of direct algebraic expressions. For the subject of linear equations, Eda offered a detailed explanation, while Tulay moved directly to the subject of linear equations without addressing the subject of linear relationships.

The data obtained as a result of the observations made by the researcher in the learning environments of Eda and Tulay’s classrooms are explained in detail with the relevant categories and codes in Table 2.

Table 2. Codes and categories of the observations made in the learning environments of Eda and Tulay’s classrooms

Category	**Code**	**Situations Observed in Eda’s Learning Environment**	**Situations Observed in Tulay’s Learning Environment**
Physical environment	**Class size**	26 students	32 students
	Seating arrangement	Students sat two to a desk, facing the board (the seats of the students were changed by the teacher during the teaching process as deemed necessary)	Students sat two to a desk, facing the board
	Technological infrastructure	There was an interactive whiteboard	There was an interactive whiteboard
Learning environment	**Teaching methods and techniques**	Usually the “recitation method “and “question and answer” method	Usually the “recitation method” and sometimes the “question and answer” method
		Alternative questions and solution methods, if any, were shared	-
		Students were taught that they must check their answers	-
		Students were taught implicit rules/formulas; the teacher guided the students to reach the rules while relying on examples and gave logical explanations of the operations she performed	Rules and formulas were introduced to the students and then examples were solved; students were not given the logical explanations underlying the rules
		Various activities were done; a painting activity was done twice during the course observations	Sometimes questions referred to as “plus questions” were asked; students who answered these questions correctly within a certain period of time were rewarded with extra points designated by a “+” symbol in the teacher’s notebook
		Group work (equation-solving activities in groups, etc.)	-
	Beginning of the lesson	The relationship of the subject with daily life was explained	-
		Students were reminded of previously learned information	Students were reminded of previously learned information
		Whether or not homework was done was checked	-
		Students were asked about the homework questions that they could not answer or had difficulty in answering, and answers were quickly provided	-
	During the lesson	A large number of examples were solved	A large number of examples were solved
		Questions were asked that included the students in the learning	Questions were asked that included the students in the learning environment:

	environment: “What does ‘first degree’ mean?” “How can you solve this equation?” “Could it be solved with a different method?” “Why was the coordinate plane needed?”	“What is an algebraic expression?” “What is an unknown?” “What is an equation?”
	Students’ questions were answered, with efforts to avoid direct answers to questions from students	Students’ questions were generally answered, but due to excessive conversation in the classroom, the teacher did not notice some students’ questions, which remained unanswered
	Students were encouraged to think adequately about the solution of the problem, usually through questions about “why” and “how”	Students were not encouraged to think adequately about the solution of the problem; the solution strategy was chosen by the teacher and shared with the students, and questions were answered quickly by the teacher, without waiting for answers from the students; this meant that students did not have the opportunity to think Questions asked by the teacher were usually questions that could be answered by saying simply “yes” or “no” and contained the answer in themselves, such as: “The difference between consecutive even numbers is 2, right? So, whatever we put on one side, we’ll put the same on the other side to keep the balance, right?”
End of the lesson	Reminders were given using the interactive board to repeat the information	-
	At the end of each lesson, worksheets were given to students as homework	-
Affective behaviors	Teacher’s communication with the students was good	Teacher’s communication with the students was good
	Students could ask questions easily, the teacher listened carefully to the questions and thoughts of the students, and feedback was given	Rarely gave feedback
	Fairness in terms of giving the students the right to speak while solving problems	Fairness in terms of choosing from among students who asked for permission to speak; however, some students did not request permission before speaking in the classroom, and no interventions were made in those situations
	Encouragement was provided in cases where students’ participation decreased, or they became anxious	No attempt was made to involve students who did not speak in class or did not pay attention to the lesson

As shown in Table 2, the physical conditions of the teachers’ classrooms were similar. The only difference was that Tulay’s class contained more students than Eda’s class. When the learning environments were examined, it was observed that both teachers usually taught by using the lecture method. In the teaching process, they both created environments where students could ask questions freely. However, in contrast to Tulay, Eda also used the question-and-answer method effectively and answered questions more sensitively. Eda asked reflective questions to help the students see their mistakes and provided opportunities for students to learn by asking questions. On the other hand, although Tulay asked the students questions, an effective discussion environment was not established as she was generally more active than the students and answered the questions she asked herself. While Eda's frequent use of "why" and "how" questions in the learning environment prompted students to think, Tulay's short questions with "yes" and "no" answers were insufficient for revealing the students' thoughts.

Eda attached particular importance to feedback, gave homework to the students after each lesson, and made an effort to address learning deficiencies by checking the homework. Another important difference was that Eda usually guided students to reach rules based on examples, whereas Tulay preferred to teach the rules and formulas to the students directly. In addition, Eda gave motivational speeches from time to time to ensure that all students participated in the lessons. For example, when she noticed the students' anxiety about algebraic expressions, she gave a speech to enhance their motivation. She increased the students' eagerness to learn by telling them that the difficulties they faced were normal and would decrease over time.

Findings from the Analysis of the Examples Presented in the Learning Environments

Findings from Examples from Eda's Classroom Environment

The analysis dimensions that Eda addressed regarding mathematical reasoning skills are shown in Table 3 for nine analyzed samples.

Table 3. Evaluation of the examples that teacher Eda presented in the teaching environment, in the sense of opportunities for mathematical reasoning

Analysis dimensions	**xample 1**	**xample 2**	**xample 3**	**xample 4**	**xample 5**	**xample 6**	**xample 7**	**xample 8**	**xample 9**
Identification of the task type	✓	×	✓	×	✓	✓	✓	✓	✓
Recognizing a Solution Method	✓	✓	✓	✓	✓	✓	✓	✓	✓
Creative Reflection	✓	✓	×	✓	×	✓	✓	✓	✓
Argumentation	✓	✓	✓	✓	×	✓	✓	✓	✓
Mathematical Foundation	✓	✓	✓	✓	✓	✓	✓	✓	✓
Alignment	×	×	×	✓	✓	✓	✓	✓	✓

Evaluation of examples that Eda presented in the classroom environment in terms of opportunities for mathematical reasoning (× indicates that the specified dimension of analysis was not addressed by the teacher; ✓ indicates that the specified dimension of analysis was addressed)

Teacher Eda also made an attempt to define the solution methods and provide the mathematical foundations in the nine examples analyzed in this study. In terms of defining the type of task, creative reflection, argumentation, and alignment, she generally made attempts, but she ignored these points for some examples. When Eda's course presentations were analyzed, it was seen that some of the examples she presented were routine problems while others were problems that the students were seeing for the first time, which could not be solved directly using a certain algorithm. In Eda's approach to presentations, mathematical features were given particular importance when examples were being offered, students participated in the solution process, and communication was at a high level. Even for routine problems, Eda asked the students questions about some parts of the problems, which was important in terms of giving students opportunities to think. When a rule was intended to be taught, Eda was careful about encouraging students to reach the rule themselves through the use of examples. This provided opportunities for students to gain high-level skills such as the abilities to think, make assumptions, and draw conclusions. Although there were deficiencies, it can be said overall that Eda created a learning environment that prioritized creativity and offered an environment for discussion, where important learning opportunities were provided to students. It can be concluded that Eda's classroom offered a learning environment where mathematical features were explained in detail and students could freely express their opinions, ask questions, and find answers to their questions. It can be concluded from the following statements that Eda encouraged students' efforts to express their ideas and provided them with opportunities to develop mathematical reasoning with creative reflection: "How else can we say it?" in the second example; "Why are you multiplying? So, how will I continue the problem? What if it asked who took 3 sugars?" in the sixth example; "Well, some points appeared on the x-axis, while some others appeared on the y-axis. So, can I make a rule accordingly?" in the seventh example; "In option B, $y = 3x$ $y = 15 + 5x$ and the last is $y = 60 - 5x$, right? How can a rule be created regarding these three, and what is the equation for the linear graphs?" in the eighth example; and "We drew two graphs. What is the difference between them?" in the ninth example.

Sample analyses of Eda's classroom environment are presented below.

Fourth Example Presented in Eda's Classroom

If x/2+1=3

find the value that x can take.

Eda: [Calls Ela to the blackboard.]
Ela: Teacher, can I do it my own way?
Eda: Will you do it from here? Do it from here [the practical method]. I want to ask you something. Which one will you move to the other side first?
Ela: I will move +1.

x/2=3+1

Students: It's wrong, teacher.
Ela: Sorry, it should be a minus.
Eda: Write the remainder on the left, and do the operation on the right.
Ela: x/2=4
Eda: What will you do for x to be alone? What is left next to the x?
Ela: Um, teacher, let me do it my own way [the way that Ela mentions is a reverse operation done based on algorithmic operations to reach the result].
Eda: Write the second part, Ela. What is next to x that must move for x to be alone?
Ela: It's 'divided by 2.'
Eda: In what form will we pass 'divided by 2'?
Ela: As 'times 2' [she tries to do the operation on the same line].

Eda: But we don't pass it on the same line. You'll make a calculation error. Write the equal sign on the same line. Let's show 2 with an arrow, it came here [to the right side of the equation]. What's left here [on the left side of the equation]?
Ela: [Writing the operation with Eda's help]:

x=4.2

x=8

Eda: Well guys, is there anything not understood?
Students: No.
Eda: Well, then let's write a problem for this equation of

x/2+1=3

Who can say one? Can you tell me, Busra?
Busra: Which number is... um... one minute... division by 2 plus 1... um... equals 3?
Eda: Yes, good! What else can we say?
Emin: Half of the number plus 1 equals 3.
Eda: Yes. Good. How else can we say it? Do we always say about one number?
Faruk: Unknown.
Baris: Of x, of a.
Eda: What else do we say?
Students: We say y.
Eda: So what else do we say verbally? For example, isn't it acceptable if we say 'Elif's age' for the unknown?
Bilal: It's possible. Do you want me to say 'Elif's age' or something else?
Eda: It's okay. Whichever you want.
Bilal: Elif's age is 1 year more than half of her brother's age. So, what is her age?
Faruk: 3 doesn't exist.
Cengiz: No, that's not true.
Bilal: No, it's wrong. Let's say an apple... But, no, it would be wrong, too.
Eda: It would be okay. Why not?
Bilal: No. 1 year more than half of Elif's age is 3. So, what is her age?
Eda: Emin, did you say 'her brother's age is 3' or something like that?
Emin: 1 year more than half of the age of Elif's brother is 3. So, what is her age?
Faruk: Isn't it the same?
Eda: Now we've said the age of Elif's brother and we've asked Elif's age. What we asked has to be the same as the unknown. It's because we find the unknown.
Emin: 1 more than half of the two apples... um...
Eda: Did you say 'of the two apples'? Then the unknown isn't acceptable. Let it be the weight of the apple. There's a bag, there's an apple in the bag, and 1 more than half of the weight of the apple in the bag equals 3 kilograms. How many kilograms does the apple in the bag weigh?
Bilal: 1 more than half of a can is 3 liters, so how many liters of water does this can hold?
Eda: Yes, it's okay.

Analysis of the Fourth Example That Eda Presented

(1) Identification of the Task Type: The problem is an equation problem. Because the teacher was only interested in solving the equation in the lesson during which this problem was solved, she did not explain the type of the problem in this example.
(2) Recognizing a Solution Method: The teacher solved the problem together with Ela, the student she called to the blackboard. She helped Ela understand what to do step by step. She kept reminding her of how to solve equations. However, she did not take into consideration Ela's repetitive question of "Can I do it my own way?" This was because Ela tended to do algorithmic operations based on four operations. She needed to shift from that mindset to abstract thinking.
(3) Creative Reflection: Eda followed the previous procedures for equation solving. However, her request that students suggest the wording of a suitable problem for the equation, as she asked after solving the equation, was intended to prompt the students to think.
(4) Argumentation: Eda was in uninterrupted communication with the students and took the steps of the operations together with them. She did not say why Ela should not use the solution method that she persistently asked to use. At that stage, no persuasive sentences were composed to convince Ela to focus on the solution method that the teacher used. It cannot be said there was a discussion at that stage. However, it is noteworthy that the teacher led a discussion about posing problems together with the students after solving the equation. She gave feedback for the ideas of all students who wanted to speak. For example, when she noticed that a student had posed a problem improperly, she had the student describe the problem again and corrected his mistake. Regarding the problem of "1 year more than half of the age of Elif's brother is 3," she told the class what the mistake was and then corrected it. At that stage, although she prevented the student from posing the problem improperly, she could have created a more effective situation with reflective questions. Questions such as "What's your mistake in your opinion?" and "How can you fix it?" could have been asked.
(5) Mathematical Foundation: Eda paid attention to mathematical notations so as to eliminate situations that would lead students to make operational errors in lines while they did mathematical operations. She constantly warned the students, saying "Make sure that equal signs come one under the other. If you move a number to the other side, show it with an arrow and write the place where it moves to on the lower line." She also intervened in wrong operations.
(6) Alignment: When Eda noticed that the students were always posing problems related to numbers in verbal form such as "of a number" or "of which number," she tried to establish alternatives to prevent such situations.

In general, Eda aimed at not only solving equations and reaching the results, but also prompting students to use their previous experiences, enabling them to understand their own operations and explain their meanings, and she worked to support students in seeing different alternatives for problems. Ensuring that students think about different alternatives creates an effect beyond the effect of doing the operations alone. Starting the reasoning process with an initially limited algorithm provided limited opportunities; however, the discussion environment and alternative questions afterwards enabled students to think creatively.

Findings from Examples from Tulay's Classroom Environment

The analysis dimensions that Tulay addressed regarding mathematical reasoning skills are shown in Table 4 for nine analyzed samples.

Table 4. Evaluation of the examples that teacher Tulay presented in the teaching environment, in the sense of opportunities for mathematical reasoning

Analysis dimensions	**Example 1**	**Example 2**	**Example 3**	**Example 4**	**Example 5**	**Example 6**	**Example 7**	**Example 8**	**Example 9**
Identification of the task type	✗	✗	✗	✗	✗	✓	✗	✓	✗
Recognizing a Solution Method	✗	✗	✗	✗	✗	✓	✗	✓	✓
Creative Reflection	✗	✗	✗	✗	✗	✗	✗	✗	✗
Argumentation	✗	✗	✗	✗	✗	✗	✓	✗	✓
Mathematical Foundation	✓	✗	✗	✗	✗	✓	✓	✓	✓
Alignment	✗	✗	✗	✗	✗	✗	✗	✗	✗

Evaluation of examples that Tulay presented in the classroom environment in terms of opportunities for mathematical reasoning (✗ indicates that the specified dimension of analysis was not addressed by the teacher; ✓ indicates that the specified dimension of analysis was addressed)

When Tulay's course presentations are analyzed, it is understood that some of the examples she presented were routine problems while some others were problems that students were seeing for the first time, which could not be solved directly using a certain algorithm. With few exceptions, Tulay adopted a presentation method in which students were generally not involved in the solution process and mathematical features received limited attention. It was observed that she asked questions about the key points in problem-solving but gave the answers quickly herself instead of waiting for answers from the students. This situation provided students with limited opportunities to learn. Tulay told them the rules while teaching the rules, but she generally did not mention the mathematical features, again with some exceptions. The mathematical foundation underlying the operations also received limited attention. In this case, a learning environment where solutions are directly given to students is an environment that paves the way for rote learning.

Fifth Example Presented in Tulay's Classroom

$3x + 6 = 2x + 8$ *Let's solve the equation.*

Tulay: Let's solve this. What will do we do? We will do away with the small x first.

$$3x + 6 = 2x + 8$$

$$3x + 6 - 2x = 2x + 8 - 2x$$

$$x + 6 = 8$$

Now we have to do away with 6. How are we going to do away with 6? Since it is +6, we need -6, right?

Students: Yes.

$$x + 6 - 6 = 8 - 6$$

$$x = 2$$

It's found like this.

Analysis of the Fifth Example That Tulay Presented

(1) Identification of the Task Type: The problem was an equation-solving problem. However, Tulay did not explain this problem type. She did not mention the difference of the equation from previously solved ones. Situations where equalities have unknowns on both sides could have been explained.

(2) Recognizing a Solution Method: No correlation was shown between the type of the task and the solution method. The method to be used for the solution was not explained. Regarding the solution, Tulay only stated: "We will do away with the small x first."

(3) Creative Reflection: Creative reflection was not observed in the solution process. The students were not involved in the solution. The teacher could have created an environment for creative reflection by directing questions in such a way as to lead the students to think. For example, she could have asked: "How we can solve this?" "Does anyone have any idea?" "Are the previous rules helpful to us?" "What can we do?"

(4) Argumentation: The teacher's solution process did not allow for any type of discussion. There was a presentation in which only algebraic operations were done and students were not involved in the solution process. A discussion environment could have been created if Tulay had first mentioned the type of the task, the solution method, and the purpose of the problem or if she explained the underlying reasons for the operations.

(5) Mathematical Foundation: The real mathematical properties of the components within the scope of the reasoning process were not mentioned. The teacher's statement that "We will do away with the small x first" was not a mathematical requirement and may have confused students. However, she did not say that this would provide students with the ease of operation or that the operation could be done in another way.

(6) Alignment: No indication was given that students could connect this with different types of problems.

In general, it was observed that Tulay asked questions from time to time while solving the problems, but she replied to the questions herself rather than waiting for answers from the students. This may have constituted an obstacle to learning the ideas of the students. If there was something a student could not fully understand, it did not become apparent at that time. In this case, Tulay's presentation did not go beyond a routine problem-solving effort in which a direct solution method was used, which provided limited opportunities for the students.

Discussion and Conclusion

When the course presentation examples of the teachers were analyzed, it was concluded that, compared to Tulay, Eda tried harder to realize the creative reflection, mathematical foundation, argumentation, and alignment components, that support reasoning. Tulay particularly made no attempts in terms of creative reflection or alignment and It was concluded that she provided limited opportunities regarding key points such as the identification of task types, recognizing solution methods, argumentation, and mathematical foundations. It should be acknowledged that both teachers had deficiencies in their teaching regarding the acquisition of mathematical reasoning skills. Indeed, (Brodie, 2010; Davidson et al., 2019, Lannin et al. 2011) mention that reasoning skills are difficult to be encouraged by teachers. Similarly, Bergvist and Lithner (2012), who indicated that teacher' presentations are based on the existing learning algorithms, usually without discussion, and while students may be given some opportunities to see the creative reasoning aspects of such thoughts and arguments based on the mathematical properties in problems, such opportunities are limited. Also, Sumpter and Hedefalk (2018) indicated that teachers have inadequacies in terms of involving children in mathematical reasoning.

In addition, Schoenfeld (1985, cited in Bergqvist & Lithner, 2012) noted that if the limitations regarding reflection, discussion, and mathematical foundation are not made obvious in teaching, the teaching process will not guide the students, it will be very difficult for students to develop mathematical skills independently, and this may cause students to think that this educational process is not important. Also, Lithner (2008), indicating that inadequacies in the learning environment considerably impact students' abilities to focus on algorithm-based reasoning. All these results reveal that learning environments need to be prepared for the acquisition of mathematical reasoning skills. The teaching approach and the way the lesson is structured are important factors in supporting students' reasoning (Oliveira & Henriques, 2021). Planning, which includes the selection of tasks as well as the identification of purposeful instructional approaches that provide students with opportunities to reason, is essential for promoting mathematical reasoning in the classroom (Herbert & Brag, 2021). In this sense, for mathematical reasoning skill-oriented instruction, how a lesson is planned by teachers, teacher actions during the lesson, and teacher-student interactions become very important. In this direction, similar situations have been addressed in studies examining teacher actions (Ellis, Ozgür, & Reiten, 2019; Mata-Pereira & da Ponte, 2017).

When the observation data were analyzed, it was seen that the teachers created a learning environment in which students could express their ideas and ask questions. However, it was concluded that there were differences in terms of the alternative questions and solution methods that the teachers posed to the students, their responses to the questions asked by the students, providing an effective question-answer interaction, and the acquisition of mathematical foundations. Teacher Eda cared about her students' encountering different types of questions and included opportunities to support higher-order thinking such as comparisons, conclusions and generalizations in the questions she posed to students. However, it was concluded that although Tülay teacher asked different questions from time to time with her answer-oriented structure, she answered the questions herself and generally adopted teaching that lacked mathematical foundations. This result is also similar to the findings of Karakus and Yesilpinar (2016), who stated that regarding the acquisition of basic field-specific skills such as problem-solving, reasoning, association, and communication, the teacher asked students to make comparisons and inferences, state similarities and differences, and solve problems in the learning process; however, the frequency of such behaviors and statements by the teacher was quite low on the whole, and there were cases where she solved the problems she gave to the students or answered the questions she asked herself. In this context, Umay and Kaf (2005) also emphasized that teachers generally focus on the correct results in problem-solving; they do not question what operations the students use or why they use those solutions, and to develop reasoning skills, it is necessary to focus on the process of the problem, not the result. From this point of view, it is thought that students will be adversely affected because directly telling students the solution methods of problems and answering questions without allowing students to think erases the student’s responsibility in the learning process (Oz and Isık, 2020). For students to gain conceptual knowledge, they need to learn in instructive classrooms that offer opportunities to engage with mathematics (Jonsson, Norqvist, Liljekvist & Lithner, 2014). When students encounter similar questions in textbooks or exams, they solve them by trying to remember procedures or algorithms that do not require conceptual understanding (Boesen, Lithner, & Palm (2010), students should be allowed to work with exercises that they cannot solve by remembering them to improve their mathematical reasoning (Birkeland, 2019). Lithner (2006) stated that focusing on algorithm-based reasoning that directs students to textbook exercises allows for short-term gains such as passing exams adapted to algorithmic reasoning, but it may also lead to long-term losses such as poor conceptual understanding and poor problem-solving abilities. Widiartana (2018) stated that to achieve the goal of maximal learning, the teacher cannot simply present the straightforward questions contained in the textbooks that are used in school; rather, it is necessary for teachers to provide open-ended mathematical problems that will serve to develop students’ reasoning abilities beyond the problems that are included in the students’ textbooks.

According to Herbert, and Bragg (2021), the question "Tell me why?" is a simple way to introduce reasoning into mathematics lessons. Because any teacher who encourages students to justify their thinking and consistently asks "Why?" is promoting reasoning. Brodie (2010) emphasizes that the key point in mathematical reasoning is the type of interaction between teachers and students in problems, the ways of encouraging students with these problems, and the types of problems to be used to encourage students. For this reason, it can be said that the type and variety of problems encountered by students and the presentation of problems are important for the development of reasoning skills.

Teacher-student interaction and the feedback given to students' answers are as important as the questions posed to students. In this regard, it was concluded that teacher Eda, unlike teacher Tulay, was sensitive about giving feedback to students and interacted with students. In order to develop mathematical reasoning, teachers' potential to encourage their students and their feedback are also necessary. Because reasoning is best learned by practicing and expressing reasoning and receiving feedback about the accuracy and appropriateness of reasoning (Bragg, Herbert, Loong, Vale, & Widjaja, 2016). The concept of feedback is crucial in the learning process; without it, students cannot know the consequences of their actions (Dahlan & Wibisono, 2021). Feedback support for reasoning can occur during whole-class discussions or during individual dialog between the student and the teacher (Smit, Hess, Taras, Bachmann, & Dober, 2023). Formative feedback to support mathematical reasoning is based on principles such as asking students to explain their thinking, challenging students to justify why their solution method will work, and challenging students to justify why their solution is correct (Olsson & Teledahl, 2018). This means that teachers should expect, validate, and reinforce the use of reasoning (Bragg et al., 2016).

Overall, this study contributes to education by emphasizing the critical role of the learning environment in the acquisition of mathematical reasoning skills. The study's conclusion that teachers often provide limited opportunities for mathematical reasoning reveals the need for improvement in teaching practices. The study also emphasizes that increasing the opportunities for mathematical reasoning in the learning environment can lead to better student achievement of conceptual learning outcomes. The reason why teachers offer limited opportunities for mathematical reasoning skills may be that they do not have detailed knowledge about mathematical reasoning skills or they do not know what can be done to acquire this skill. The teaching opportunities that students encounter in the classroom are limited to the opportunities that teachers offer to students. From this point of view, teachers should be encouraged to include more comprehensive elements in their presentations about the characteristics of problems or alternative solution methods, which allow for discussion and stimulate students' reasoning. Since it is difficult for teachers to develop themselves in this sense, teachers should be provided with the necessary support such as time and in-service training on how to gain reasoning skills. In addition, since the findings of our study are limited to two teachers and the algebra learning domain, more comprehensive studies are needed both in terms of sampling and depth. The differences in the teachers' presentations suggest that the mathematical reasoning skill competencies of the students of these teachers may also be different. With this prediction, studies on whether teachers' presentations affect the types of mathematical reasoning exhibited by students can also be planned.

Note

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